Response of Linear Periodically Time Varying Systems to Random Excitation

D. A. PRELEWICZ*
Washington University, St. Louis, Mo.

Introduction

RECENTLY developed stochastic models for the flapping vibration of helicopter rotor blades[1] give rise to the problem of determining the response of periodically time varying linear systems to random excitation. Mathematically, the problem is to determine the statistical properties of a response process $\bar{x}(t)$ satisfying

$$d\bar{x}/dt = A(t)\bar{x} + \bar{f}; \qquad A(t) = A(t+T) \tag{1}$$

where $\bar{f}(t)$ is a random input process and the matrix $A(t)$ is given in terms of system parameters.

In this Note, the class of weakly periodic nonstationary processes is shown to arise in a natural way in connection with the system (1). In particular, it is shown that the steady-state response of the system (1), if it exists, to a weakly periodic nonstationary process with period T is also a weakly periodic nonstationary process with period T.

In Refs. 1–4 both time domain and mixed time frequency domain methods have been used to calculate response statistics for Eqs. (1). These relations are generalized within the framework of weakly periodic nonstationary processes.

Deterministic Case

A solution of Eqs. (1) for $t > t_0$ is (Ref. 5, p. 97)

$$\bar{x}(t) = Z(t,t_0)\bar{x}(t_0) + \int_{t_0}^{t} Z(t,s)\bar{f}(s)\,ds \tag{2}$$

where $Z(t,t_0)$ is the transition matrix satisfying

$$dZ/dt = A(t)Z, \quad t > t_0 \tag{3a}$$

$$Z(t_0,t_0) = I \text{ (identity matrix)} \tag{3b}$$

Furthermore, the periodicity of $A(t)$ implies that (Ref. 5, p. 106)

$$Z(t,t_0) = P(t)e^{C(t-t_0)}P^{-1}(t_0), \quad t > t_0 \tag{4}$$

where C is a constant matrix and $P(t+T) = P(t)$. The steady-state response is given by

$$\bar{x}(t) = \int_{-\infty}^{t} Z(t,s)\bar{f}(s)ds \tag{5}$$

Random Excitation

Letting $u = t - s$ in Eq. (5), using Eq. (4) and taking expectations gives the following expressions for the steady-state mean and autocorrelation of the output process:

$$E\{\bar{x}(t)\} = \bar{\mu}_x(t) = \int_0^{\infty} P(t)e^{Cu}P^{-1}(t-u)\bar{\mu}_f(t-u)du \tag{6}$$

$$E\{\bar{x}(t_1)\bar{x}^T(t_2)\} = R_{xx}(t_1,t_2) = \int_0^{\infty}\int_0^{\infty} P(t_1)e^{Cu_1}P^{-1}(t_1-u_1) \tag{7}$$

$$R_{ff}(t_1-u_1,t_2-u_2)P^{-1T}(t_2-u_2)e^{C^Tu_2}P^T(t_2)du_1du_2$$

Suppose that $\bar{f}(t)$ exists in a mean square sense and that

$$\bar{\mu}_f(t) = \bar{\mu}_f(t+T) \tag{8}$$

$$R_{ff}(t_1,t_2) = R_{ff}(t_1+T,t_2+T) \tag{9}$$

Such a process is said to be weakly period nonstationary with period T. It is precisely these processes which are of interest in the rotor vibration problem. From Eq. (6), (7), and the periodicity of $P(t)$ it is apparent that if the steady-state response exists then it is also weakly periodic nonstationary of period T. Hence linear periodically time varying systems with period T "preserve" weakly periodic nonstationary processes of period T in the same sense that linear time invariant systems "preserve" weakly stationary processes.

The periodicity can now be used to simplify the calculation of the mean and the autocorrelation. From Eq. (2)

$$\bar{\mu}_x(t) = Z(t,t_0)\bar{\mu}_x(t_o) + \int_{t_0}^{t} Z(t,s)\bar{\mu}_f(s)\,ds \tag{10}$$

Using the periodicity of $\bar{\mu}_x(t_0)$

$$\bar{\mu}_x(t_0) = \bar{\mu}_x(t_0+T) = Z(t_0+T,t_0)\bar{\mu}_x(t_0) + \int_{t_0}^{t_0+T} Z(t_0+T,s)\bar{\mu}_f(s)ds \tag{11}$$

Solving for $\bar{\mu}_x(t_0)$ gives

$$\bar{\mu}_x(t_0) = [I - Z(t_0+T,t_0)]^{-1}\int_{t_0}^{t_0+T} Z(t_0+T,s)\bar{\mu}_f(s)ds \tag{12}$$

Now from Eq. (2) we obtain

$$R_{xx}(t_1+T,t_2+T) = Z(t_1+T,t_1)R_{xx}(t_1,t_2+T) + \int_{t_1}^{t_1+T} Z(t_1+T,s_1)R_{fx}(s_1,t_2+T)ds_1 \tag{13}$$

$$R_{xx}(t_1,t_2+T) = R_{xx}(t_1,t_2)Z^T(t_2+T,t_2) + \int_{t_2}^{t_2+T} R_{xf}(t_1,s_2)Z^T(t_2+T,s_2)ds_2 \tag{14}$$

Substituting Eq. (14) into Eq. (13) and using the periodicity of $R_{xx}(t_1,t_2)$ gives

$$Z^{-1}(t_1+T,t_1)R_{xx}(t_1,t_2) - R_{xx}(t_1,t_2)Z^T(t_2+T,t_2) = \int_{t_2}^{t_2+T} R_{xf}(t_1,s_2)Z^T(t_2+T,s_2)ds_2 \tag{15}$$

$$+ \int_{t_1}^{t_1+T} Z^{-1}(t_1+T,t_1)Z(t_1+T,s_1)R_{fx}(s_1,t_2+T)ds_1$$

where in the steady state

$$R_{xf}(t_1,s_2) = \int_{-\infty}^{t_1} Z(t_1,s_1)R_{ff}(s_1,s_2)ds_1 \tag{16a}$$

$$R_{fx}(s_1,t_2+T) = \int_{-\infty}^{t_2+T} R_{ff}(s_1,s_2)Z^T(t_2+T,s_2)ds_2 \tag{16b}$$

Combining Eqs. (15) and (16)

$$Z^{-1}(t_1+T,t_1)R_{xx}(t_1,t_2) - R_{xx}(t_1,t_2)Z^T(t_2+T,t_2) = \int_{t_2}^{t_2+T}\int_{-\infty}^{t_1} Z(t_1,s_1)R_{ff}(s_1,s_2)Z^T(t_2+T,s_2)ds_1ds_2 \tag{17}$$

$$+ \int_{t_1}^{t_1+T}\int_{-\infty}^{t_2+T} Z(t_1,s_1)R_{ff}(s_1,s_2)Z^T(t_2+T,s_2)ds_2ds_1$$

where $Z(t_1,s_1)$ in the second integral on the right is

$$Z(t_1,s_1) = Z^{-1}(t_1+T,t_1)Z(t_1+T,s_1) = P(t_1)e^{C(t_1-s_1)}P^{-1}(s_1) \tag{18}$$

It is convenient to use Eq. (17) to obtain the instantaneous autocorrelation matrix $R_{xx}(t,t)$ and to determine $R_{xx}(t_1,t_2)$ from

$$R_{xx}(t_1,t_2) = Z(t_1,t_2)R_{xx}(t_2,t_2) + \int_{t_2}^{t_1}\int_{-\infty}^{t_2} Z(t_1,s_1)R_{ff}(s_1,s_2) Z^T(t_2,s_2)ds_2ds_1, \quad t_1 > t_2 \tag{19}$$

obtained from Eq. (2) by letting $t_0 = t_2$, postmultiplying by $\bar{x}^T(t_2)$, taking the expectation and then using Eq. (16b). This approach gives an explicit expression for $R_{xx}(t_1,t_2)$.

In the variables $\tau = t_1 - t_2$ and $\zeta = t_1 + t_2$ it is apparent that, for fixed τ, $R_{xx}(\tau,\zeta)$ is periodic in ζ with period $2T$. Also, using Eq. (7) it is easy to show that

$$R_{xx}(\tau,\zeta) = R_{xx}{}^T(-\tau,\zeta) \tag{20}$$

Received March 16, 1972.
Index category: VTOL Vibration.
* Assistant Professor. Associate Member AIAA.

Hence $R_{xx}(\tau, \zeta)$ is completely determined if it is known on a a strip $0 \leqq \tau < \infty, t \leqq \zeta < t + 2T$.

In the special case of weakly periodic nonstationary white noise excitation, i.e.,

$$R_{ff}(t_1, t_2) = F(t_1)\delta(t_1 - t_2) \tag{21}$$

where $F(t_1) = F(t_1 + T)$ and $\delta(\ )$ is the Dirac delta function, Eq. (17) gives

$$Z^{-1}(t + T, t)R_{xx}(t, t) - R_{xx}(t, t)Z^T(t + T, t) = \int_t^{t+T} Z(t, s)F(s)Z^T(t + T, s)ds \tag{22}$$

and Eq. (19) reduces to

$$R_{xx}(t_1, t_2) = Z(t_1, t_2)R_{xx}(t_2, t_2), t_1 > t_2 \tag{23}$$

For the white noise case, a differential equation satisfied by $R_{xx}(t, t)$ is given in Refs. 3 and 4. Equation (22) gives the periodic solution of that equation. Also, the preceding analysis shows that the direct time domain approach is not limited to white noise excitation.

Spectral Density

$R_{ff}(\tau, \zeta)$ is, for fixed τ, periodic in ζ with period $2T$. Hence, if $R_{ff}(\tau, \zeta)$ is absolutely integrable in τ, then it is apparent that $R_{ff}(\tau, \zeta)$ has the spectral representation

$$R_{ff}(\tau, \zeta) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ff}(\omega, n)e^{i\omega\tau}e^{in\pi\zeta/T}d\omega \tag{24}$$

Inversion yields

$$\Phi_{ff}(\omega, n) = \frac{1}{4\pi T}\int_{-\infty}^{\infty}\int_{-T}^{T} R_{ff}(\tau, \zeta)e^{-i\omega\tau}e^{-in\pi\zeta/T}d\zeta d\tau \tag{25}$$

Substituting Eq. (24) into Eq. (7), using Eq. (4) and rearranging terms gives

$$R_{xx}(t_1, t_2) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} H(t_1, \frac{n\pi}{T} - \omega)\Phi_{ff}(\omega, n)H^T(t_2, \frac{n\pi}{T} + \omega)\, d\omega \tag{26}$$

where

$$H(t, \lambda) = \int_{-\infty}^{t} Z(t, s)e^{i\lambda s}ds \tag{27}$$

Equation (26) is a generalization of the mixed time frequency relation used in Ref. 1 to include weakly periodic nonstationary excitation.

If $R_{ff}(t_1, t_2)$ can be expressed in the product form

$$R_{ff}(t_1, t_2) = F(t_1)R(t_1 - t_2)G^T(t_2) \tag{28}$$

where $R(t_1 - t_2)$ has a Fourier transform $\Phi(\omega)$, then

$$R_{xx}(t_1, t_2) = \int_{-\infty}^{\infty} H_1(t_1, -\omega)\Phi(\omega)H_2^T(t_2, \omega)d\omega \tag{29}$$

where

$$H_1(t, \lambda) = \int_{-\infty}^{t} Z(t, s)F(s)e^{i\lambda s}ds \tag{30}$$

$$H_2(t, \lambda) = \int_{-\infty}^{t} Z(t, s)G(s)e^{i\lambda s}ds$$

Excitation satisfying Eq. (28) is treated in Ref. 3.

It is interesting to note that a direct relationship exists between the spectral densities of the input and output processes. First notice that for fixed τ, $Z(\tau, \zeta)$ is periodic in ζ with period $2T$. Hence assuming that $Z(\tau, \zeta)$ is absolutely integrable in τ, it has the same type of spectral representation as $R_{ff}(\tau, \zeta)$ namely

$$Z(\tau, \zeta) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_z(\omega, n)e^{i\omega\tau}e^{in\pi\zeta/T}d\omega \tag{31}$$

Inverting gives

$$\Phi_z(\omega, n) = \frac{1}{4\pi T}\int_{-\infty}^{\infty}\int_{-T}^{T} Z(\zeta, \tau)e^{-i\omega\tau}e^{-in\pi\zeta/T}\, d\zeta\, d\tau \tag{32}$$

Substituting Eq. (31) into Eq. (7), changing variables t_1, t_2 to τ, ζ, and simplifying using the appropriate Fourier orthogonality relations yields

$$R_{xx}(\tau, \zeta) = 4\pi^2 \sum_{j,k,l=-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_z[\lambda + (l + j)(\pi/T), j]\Phi_{ff}(\lambda, l) \tag{33}$$

$$\Phi_z^T[-\lambda + (l + k)\pi/T, k]\, e^{i\tau[(j-k)\pi/T+\lambda]}e^{i\zeta(j+k+l)\pi/T}d\lambda$$

Now substituting Eq. (33) into the expression for $\Phi_{xx}(\omega, n)$ [Eq. (24) with f replaced by x] yields after simplification

$$\Phi_{xx}(\omega, n) = 4\pi^2 \sum_{j,k=-\infty}^{\infty} \Phi_z\left[\omega + (n - j)\frac{\pi}{T}, j\right]$$

$$\Phi_{ff}\left[\omega + (k - j)\frac{\pi}{T}, n - k - j\right]\Phi_z^T\left[-\omega + (n - k)\frac{\pi}{T}, k\right] \tag{34}$$

Conclusions

When viewed within the framework of weakly periodic nonstationary processes, the steady-state analysis of the random response of periodically time varying systems is quite similar to the well known analysis of time invariant systems subject to weakly stationary excitation. That is, all of the results derived herein reduce to well known results for linear time invariant systems subject to weakly stationary excitation. These results should prove useful in the further development of rotor vibration models.

References

[1] Gaonkar, G. H. and Hohenemser, K. H., "Stochastic Properties of Turbulence Excited Rotor Blade Vibrations," *AIAA Journal*, Vol. 9, No. 3, March 1971, pp. 419–424.

[2] Gaonkar, G. H. and Hohenemser, K. H., "Comparison of Two Stochastic Models for Threshold Crossing Studies of Rotor Blade Flapping Vibrations," AIAA Paper 71-389, Anaheim, Calif., 1971.

[3] Gaonkar, G. H. and Hohenemser, K. H., "An Advanced Stochastic Model for Threshold Crossing Studies of Rotor Blade Vibrations," *AIAA Journal*, Vol. 10, No. 8, Aug. 1972, pp. 1100–1101.

[4] Wan, F. Y. M. and Lakshmikantham, C., "Rotor Blade Response to Random Loads: A Direct Time Domain Approach," AIAA Paper 72-169, San Diego, Calif., 1972.

[5] Struble, R. A., *Nonlinear Differential Equations*, McGraw-Hill, New York, 1962.

Simplified Conservation Laws for Finite-Difference Computations

Bruce Masson*
Dynamic Science Division, Ultrasystems Inc., Irvine, Calif.

1. Introduction

FINITE-DIFFERENCE methods have proven to be powerful techniques for the solution of many fluid dynamical problems, particularly inviscid flows containing embedded shock waves. If the difference method is cast in conservation form, then the same difference equations can be applied at all grid points, including those at which shock waves are present. Although the interior structure given to the shock is artificial, the conservation form of the equations guarantees that the correct shock jump conditions are satisfied.

Received March 20, 1972. This work supported by U. S. Army Munitions Command, Picatinny Arsenal, Dover, N. J. under Contract DAAA21-72-C-0334.

Index categories: Subsonic and Transonic Flow; Nonsteady Aerodynamics; Computer Technology and Computer Simulation Techniques.

* Member Technical Staff. Member AIAA.